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Infinitely many conservation laws for the discrete KdV equation

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Abstract

Rasin and Hydon (2007 *J. Phys. A: Math. Theor.* **40** 12763–73) suggested a way to construct an infinite number of conservation laws for the discrete KdV equation (dKdV), by repeated application of a certain symmetry to a known conservation law. It was not decided, however, whether the resulting conservation laws were distinct and nontrivial. In this paper we obtain the following results: (1) we give an alternative method to construct an infinite number of conservation laws using a discrete version of the Gardner transformation. (2) We give a direct proof that the conservation laws obtained by the method of Rasin and Hydon are indeed distinct and nontrivial. (3) We consider a continuum limit in which the dKdV equation becomes a first-order eikonal equation. In this limit the two sets of conservation laws become the same, and are evidently distinct and nontrivial. This proves the nontriviality of the conservation laws constructed by the Gardner method, and gives an alternative proof of the nontriviality of the conservation laws constructed by the method of Rasin and Hydon.

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1. Introduction

The theory of integrable quad-graph equations starts from the works of Hirota [13–15], where the author presents a nonlinear partial difference equation (PΔE) which 'reduces to the Korteweg-de Vries equation in the weakly nonlinear and continuum limit' [13]. More recently many interesting properties of quad-graph equations have been found that can be interpreted as integrability properties [34]. Lax pairs for some quad-graph equations were presented in [3, 21]. The ultra-local singularity confinement criterion for quad-graphs equations was developed in [11, 12, 24]. This criterion can be viewed as a discrete version of the Painlevé property. Further research showed that certain quad-graph equations are consistent on the cube and all equations satisfying this condition were classified [1, 4]. From this consistency one can derive the Lax pair and Bäcklund transformations. So consistency on the cube also becomes an integrability criterion, even though there is no analog of this in the theory of partial differential equations (PDE). The derivation of master symmetries for certain quad-graph equations can be found in [29]. The master symmetry can be used to construct infinite hierarchies of symmetries, and this can also can be interpreted as an integrability property.

One more criterion for integrability is the existence of an infinite number of conservation laws (in involution, in the case of a Hamiltonian system). The investigation of conservation laws of quad-graph equations was initiated by Hydon in [17]. There the author presented a method for computation of conservation laws of quad-graph equations and derived conservation laws for the modified discrete Korteweg-de Vries (mdKdV) equation. Hydon's method was improved in [26–28]; in these papers the authors also derived three and five point conservation laws for all the equations in the ABS classification. In [28] a suggestion for constructing an infinite number of conservation laws was given (for many of the equations in the ABS classification). But it was not shown that all the resulting conservation laws were distinct and nontrivial.

In this paper we focus on conservation laws for the discrete KdV equation (dKdV, **H1** in the ABS classification)

$$(u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + \beta - \alpha = 0. \tag{1}$$

Here $k, l \in \mathbb{Z}$ are independent variables and $u_{0,0} = u(k,l)$ is a dependent variable that is defined on the domain $\mathbb{Z}^2$. We denote the values of this variable at other points by $u_{i,j} = u(k+i, l+j) = S_k^i S_l^j u_{0,0}$, where S_k, S_l are the unit forward shift operators in k and l, respectively. In [28] it was shown that by applying the symmetry

$$X = \frac{k}{u_{1,0} - u_{-1,0}} \frac{\partial}{\partial u_{0,0}} - \partial_\alpha$$

to the dKdV conservation law

$$F = -\ln(u_{0,1} - u_{-1,0}), \qquad G = \ln(u_{1,0} - u_{-1,0}),$$

and then adding a trivial conservation law, we obtain a new nontrivial dKdV conservation law

$$F_{\text{new}} = \frac{-1}{(u_{0,0} - u_{-2,0})(u_{0,1} - u_{-1,0})}, \qquad G_{\text{new}} = \frac{1}{(u_{0,0} - u_{-2,0})(u_{1,0} - u_{-1,0})}.$$

It was suggested that an infinite number of conservation laws could be generated by repeating this procedure. We call this method of constructing conservation laws *the symmetry method*.

The first result of this paper is an alternative method to construct an infinite number of conservation laws using a discrete version of the *Gardner transformation*. The Gardner transformation is an elementary method to construct the infinite number of conservation laws of the continuum KdV equation [7, 20]. We believe that the conservation laws of dKdV obtained from this new method are the same as those obtained by the symmetry method, but do not prove it. Our second result is a direct proof of the nontriviality of the conservation laws obtained by the symmetry method. The proof exploits a fundamental lemma about conservation laws of a particular form as well as properties of the discrete Euler operator. Our third contribution is to consider a certain continuum limit of dKdV, in which the equation becomes a first-order eikonal equation. In this limit the two sets of conservation laws become the same, and are evidently nontrivial. This proves the nontriviality of the conservation laws constructed by the Gardner method, and gives an alternate proof of nontriviality of the conservation laws constructed by the symmetry method. It also provides evidence for our hypothesis that the conservation laws constructed by the two methods do indeed coincide.

The structure of this paper is as follows: section 2 is a summary of the general theory of conservation laws, including a new lemma about a particularly significant kind of conservation laws for P∆Es. Section 3 presents the Gardner method for dKdV. Section 4 gives the proof of nontriviality of the conservation laws obtained by the symmetry method. In section 5 we consider a continuum limit. Finally, section 6 contains some concluding comments and questions for further study.

2. Conservation laws

We find it useful to summarize the standard results on conservation laws for both PDEs and P∆Es, as general background to the paper, and in particular as background for a crucial lemma we will need for section 4.

For a scalar partial differential equation with two independent variables x, t and a single dependent variable u, a (local) conservation law is an expression of the form

$$\partial_t G + \partial_x F = 0,$$

which holds as a consequence of the equation. Here F, G, which are called 'the components of the conservation law', are functions of x, t, u and a finite number of partial derivatives of u. For example, if u satisfies the KdV equation

$$u_t = \tfrac{1}{4}u_{xxx} + 3uu_x$$

we then have

$$\partial_t(u) + \partial_x\left(-\tfrac{1}{4}u_{xx} - \tfrac{3}{2}u^2\right) = 0,$$
$$\partial_t(u^2) + \partial_x\left(-\tfrac{1}{2}uu_{xx} + \tfrac{1}{4}u_x^2 - 2u^3\right) = 0,$$
$$\partial_t\left(4u^3 - u_x^2\right) + \partial_x\left(-9u^4 + \tfrac{1}{2}u_xu_{xxx} - \tfrac{1}{4}u_{xx}^2 - 3u^2u_{xx} + 6uu_x^2\right) = 0.$$

We say a conservation law is trivial for an equation if by application of the equation to the individual components of the conservation law we can bring them into a form for which the law holds for all functions u, not just on solutions of the equation. Equivalently, we say the conservation law with components F, G is trivial if we can write

$$F = F_0 - \partial_t f \qquad G = G_0 + \partial_x f,$$

where F_0, G_0 both vanish as a consequence of the equation and f is an arbitrary function of x, t, u and a finite number of partial derivatives of u [22]. For KdV, and more generally for any equation of the form $u_t = p(x, t, u, u_x, u_{xx}, \ldots)$, there is a simple way to recognize nontrivial conservation laws. We first use the equation to eliminate all occurrences of t derivatives in G. The conservation law is trivial if and only if the resulting component G is a (total) x-derivative (of some function f of x, t, u and a finite number of x-derivatives of u). Below we will prove an analog of this result for dKdV.

Moving now to the discrete case, we consider a general quad-graph equation

$$P(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, \mathbf{a}) = 0, \tag{2}$$

where $\mathbf{a}$ denotes a vector of parameters. A conservation law is an expression of the form

$$(S_l - I)G + (S_k - I)F = 0, \tag{3}$$

which holds as a consequence of the equation. F and G are called 'the components of the conservation law', and are functions of k, l, the parameters $\mathbf{a}$, and the values of the variable u at a finite number of points. In the above I denotes the identity mapping.

We say a conservation law is trivial for an equation if it takes the form

$$F = F_0 - (S_l - I)f \qquad G = G_0 + (S_k - I)f, \tag{4}$$

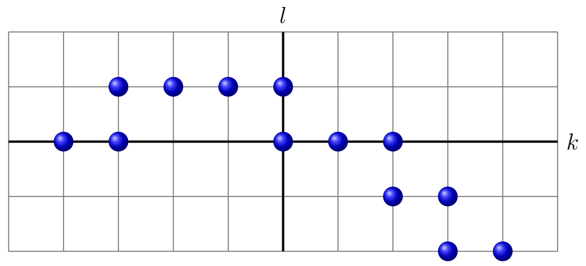

Figure 1. Possible points of G_0.

where F_0, G_0 both vanish as a consequence of the equation and f is an arbitrary function of k, l, **a** and the values of the variable u at a finite number of points.

The conservation laws we will study in this paper for dKdV have a special form:

Definition 1. *We say the conservation law (3) for the quad-graph equation (2) is 'on the horizontal line' if G depends only on values of $u_{n,m}$ with $m = 0$, and 'on the vertical line' if F depends only on values of $u_{n,m}$ with $n = 0$.*

We then have the following lemma:

Lemma 1. *For the dKdV equation, a trivial conservation law on the horizontal (vertical) line can always be presented in a form with $G = (S_k - I)f$ $(F = (S_l - I)f)$, where f is a function on the horizontal (vertical) line.*

This is a discrete analog of the usual way to recognize trivial conservation laws for the continuum KdV equation, as described above. Even though we state the lemma here just for the dKdV equation, it in fact holds for other quad-graph equations; the equation is only used in the proof in a weak way. Note that if the G of the trivial conservation law on the horizontal line in the lemma depends on $u_{n,0}$ with $n_l \leqslant n \leqslant n_r$, then f depends on $u_{n,0}$ with $n_l \leqslant n \leqslant n_r - 1$ (and similarly in the vertical case). We present the proof just for the horizontal case, the vertical case is similar.

Proof. Since the conservation law is trivial, we can write $G = G_0 + (S_k - I)f$, where G_0 vanishes on solutions of the equation, but it is possible that G_0 and f may not depend only on values on the line, see figure 1. The forms of G_0 and f are not uniquely determined; we have the freedom to add to f any term that vanishes on solutions of the equation and subtract a corresponding term from G_0. Using the equation in the form

$$u_{1,1} = \omega_2(u_{0,0}, u_{0,1}, u_{1,0}), \tag{5}$$

we can eliminate the occurrence in f of all values of u above the horizontal line (where G is valued), with the possible exception of values on a vertical line going through the leftmost point on the horizontal line. Similarly, using the equation in the form

$$u_{1,0} = \omega_1(u_{0,0}, u_{0,1}, u_{1,1}) \tag{6}$$

we can eliminate in f values of u below the horizontal line and to the right of the vertical line. Thus without loss of generality we can take f to be valued on a 'tetris', see figure 2. More formally, we have

$$f = f\left(u_{n_l,m_b}, u_{n_l,m_b+1}, \ldots, u_{n_l,m_t}, u_{n_1+1,0}, u_{n_l+2,0}, \ldots, u_{n_r,0}\right).$$

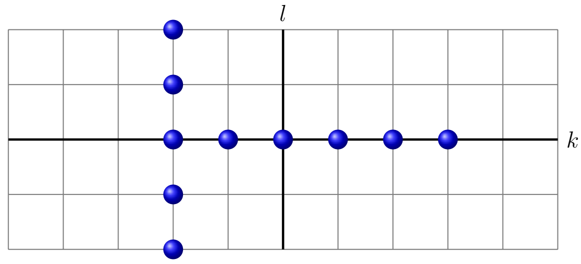

Figure 2. Points of f (after moving to a tetris).

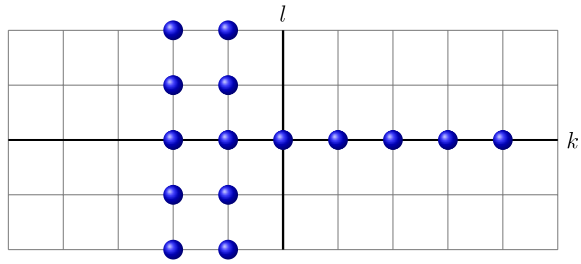

Figure 3. Points of $(S_k - I)f$ (after moving to a tetris).

Here $m_b \leqslant 0$ and $m_t \geqslant 0$ denote the lowest and highest values on the vertical axis, and n_l, n_r the lowest and highest on the horizontal axis. The corresponding points of $(S_k - I)f$ are indicated in figure 3. We know that $G = G_0 + (S_k - I)f$ where G is valued on the horizontal line and $(S_k - I)f$ is valued on the points shown in figure 3. Assume now that $m_t > 0$, i.e. that f depends nontrivially on a point above the line. We write

$$
\begin{aligned}
(S_k - I)f = {} & f\left(u_{n_l+1,m_b}, u_{n_l+1,m_b+1}, \ldots, u_{n_l+1,m_t}, u_{n_1+2,0}, u_{n_l+3,0}, \ldots, u_{n_r+1,0}\right) \\
& - f\left(u_{n_l,m_b}, u_{n_l,m_b+1}, \ldots, u_{n_l,m_t}, u_{n_1+1,0}, u_{n_l+2,0}, \ldots, u_{n_r,0}\right) \\
= {} & \big[f\left(u_{n_l+1,m_b}, u_{n_l+1,m_b+1}, \ldots, u_{n_l+1,m_t}, u_{n_1+2,0}, u_{n_l+3,0}, \ldots, u_{n_r+1,0}\right) \\
& - f\big(u_{n_l+1,m_b}, u_{n_l+1,m_b+1}, \ldots, \omega_2(u_{n_l,m_t-1}, u_{n_l,m_t}, u_{n_1+1,m_t-1}), \\
& u_{n_1+2,0}, u_{n_l+3,0}, \ldots, u_{n_r+1,0}\big)\big] \\
& + \big[f\big(u_{n_l+1,m_b}, u_{n_l+1,m_b+1}, \ldots, \omega_2(u_{n_l,m_t-1}, u_{n_l,m_t}, u_{n_1+1,m_t-1}), \\
& u_{n_1+2,0}, u_{n_l+3,0}, \ldots, u_{n_r+1,0}\big) \\
& - f\big(u_{n_l,m_b}, u_{n_l,m_b+1}, \ldots, u_{n_l,m_t}, u_{n_1+1,0}, u_{n_l+2,0}, \ldots, u_{n_r,0}\big)\big].
\end{aligned}
$$

The term in the first square brackets here vanishes on solutions of the equation. The term in the second square brackets only depends on a single value at the vertical level m_t, u_{n_l,m_t}. Evidently, if we want $G = G_0 + (S_k - I)f$ to hold, where G is valued on the line and G_0 vanishes on solutions of the equation, we must demand that the term in the second square brackets in fact be independent of u_{n_l,m_t} on solutions of the equation, i.e. that

$$
\begin{aligned}
& \left.\frac{\partial f}{\partial u_{n_l,m_t}}\right|_{(u_{n_l+1,m_b}, u_{n_l+1,m_b+1}, \ldots, \omega_2, u_{n_1+2,0}, u_{n_l+3,0}, \ldots, u_{n_r+1,0})} \left.\frac{\partial \omega_2}{\partial u_{0,1}}\right|_{(u_{n_l,m_t-1}, u_{n_l,m_t}, u_{n_1+1,m_t-1})} \\
& \quad = \left.\frac{\partial f}{\partial u_{n_l,m_t}}\right|_{(u_{n_l,m_b}, u_{n_l,m_b+1}, \ldots, u_{n_l,m_t}, u_{n_1+1,0}, u_{n_l+2,0}, \ldots, u_{n_r,0})}
\end{aligned}
$$

on solutions of the equation. (On the LHS of the above formula ω_2 is written as short for $\omega_2(u_{n_l,m_t-1}, u_{n_l,m_t}, u_{n_1+1,m_t-1})$.) Clearly the necessary identity will hold if $\frac{\partial f}{\partial u_{n_l,m_t}} = 0$, i.e. if f is independent of u_{n_l,m_t}. Further manipulation (using the equation to make the two sides of the last equation depend on the same values of u) shows that indeed this is the only case. Thus we have a contradiction, our assumption that f depends nontrivially on a point above the line has been proved wrong.

By a similar argument we show that f does not depend on points below the line. Finally, if $G = G_0 + (S_k - I)f$ and both G and f are on the line, and G_0 vanishes on solutions of the equation, clearly we must have $G_0 = 0$ and the lemma is established. □

Having established this lemma (at least for the case of the dKdV equation, and it is also true in some generality), it just remains to indicate how this allows us to easily identify nontrivial conservations laws on the line. For this we use the discrete Euler operator. The discrete Euler operator is defined [18] by

$$E(A) = \sum_{n,m} S_k^{-n} S_l^{-m} \left(\frac{\partial A}{\partial u_{n,m}} \right). \tag{7}$$

Here A is a function of finitely many values $u_{n,m}$ of the variable u. Clearly

$$E(S_k A) = E(S_l A) = E(A),$$

and thus

$$E((S_k - I)A) = E((S_l - I)A) = 0.$$

So given the G (F) component of a conservation law on the horizontal (vertical) line, if application of the Euler operator does not give zero, it is nontrivial.

3. The Gardner method for dKdV

Before presenting the Gardner method for generating conservation laws of dKdV, we review the method for continuum KdV. The Gardner method for KdV starts with the Bäcklund transformation. The Bäcklund transformation states that if u solves KdV then so does $u + v_x$ where v is a solution of the system

$$\begin{aligned} v_x &= \theta - 2u - v^2 \\ v_t &= -\tfrac{1}{2}u_{xx} + (\theta + u)(\theta - 2u - v^2) + u_x v. \end{aligned}$$

It is straightforward to check that if u solves KdV then these two equations for v are consistent (i.e.,$(v_x)_t = (v_t)_x$) and also that if v is defined by these two equations then $u + v_x$ does indeed satisfy KdV. Here θ is a parameter. The next thing to do is to observe that if we could solve the first equation of the Bäcklund transformation to write v as a function of u and (a finite number of) its x-derivatives, then we would have the following conservation law:

$$\partial_t v + \partial_x \left(\tfrac{1}{2}u_x - (u + \theta)v \right) = 0.$$

This cannot be done explicitly, but it is possible to write a formal solution of the first equation of the Bäcklund transformation to give v in terms of u as a formal series in decreasing powers of $\theta^{1/2}$. The first few terms of the relevant series are

$$v = \theta^{1/2} - \frac{u}{\theta^{1/2}} + \frac{u_x}{2\theta} - \frac{u_{xx} + 2u^2}{4\theta^{3/2}} + \frac{u_{xxx} + 8uu_x}{8\theta^2} - \frac{u_{xxxx} + 8u^3 + 10u_x^2 + 12uu_{xx}}{16\theta^{5/2}} + O(\theta^{-3}).$$

Each coefficient in this expansion gives (the G component of) a conservation law. More precisely, the coefficients of integer powers of θ give trivial conservation laws, and the coefficients of half-integer powers give the 'G' components of nontrivial conservation laws. The 'F' components can be found from the corresponding coefficient in the expansion of $\frac{1}{2}u_x - (u+\theta)v$. Examining in detail the way in which the terms of the above series are generated, it can be shown that the coefficient of $\theta^{-n+1/2}$ has a term proportional to u^n, for $n = 1, 2, 3, \ldots$ and thus the corresponding conservation law is nontrivial [7, 20].

We now try to reproduce this for dKdV. The Bäcklund transformation for dKdV [2, 31] is $u \to \tilde{u}$ where

$$(\tilde{u}_{0,0} - u_{0,1})(u_{0,0} - \tilde{u}_{0,1}) = \theta - \beta,$$
$$(\tilde{u}_{0,0} - u_{1,0})(u_{0,0} - \tilde{u}_{1,0}) = \theta - \alpha.$$

Here θ is a parameter. Once again, by a Bäcklund transformation we mean two things: that the above equations for $\tilde{u}$ are consistent if u satisfies dKdV, and that $\tilde{u}$ defined by these equations also satisfies dKdV. As in the case of continuum KdV, we cannot in general solve the equations of the Bäcklund transformation to write $\tilde{u}$ in terms of u. However there are several special cases. In the case $\theta = \beta$ we can take $\tilde{u}_{0,0} = u_{0,1}$ or $u_{0,-1}$, and in the case $\theta = \alpha$ we can take $\tilde{u}_{0,0} = u_{1,0}$ or $u_{-1,0}$. For θ near these special values we can find series solutions. Consider the case $\theta = \alpha + \epsilon$ where ϵ is small, and look for a solution of the Bäcklund transformation in the form

$$\tilde{u}_{0,0} = u_{1,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i.$$

We just look at the second equation of the Bäcklund transformation. This reads

$$\epsilon = \left(\sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i \right) \left(u_{0,0} - u_{2,0} - \sum_{i=1}^{\infty} v_{1,0}^{(i)} \epsilon^i \right).$$

The leading order approximation gives

$$v_{0,0}^{(1)} = \frac{1}{u_{0,0} - u_{2,0}}. \tag{8}$$

Higher order terms give

$$v_{0,0}^{(i)} = \frac{1}{u_{0,0} - u_{2,0}} \sum_{j=1}^{i-1} v_{0,0}^{(j)} v_{1,0}^{(i-j)}, \qquad i = 2, 3, \ldots. \tag{9}$$

We note that all these formulae are on the horizontal line, i.e. all the $v_{0,0}^{(i)}$ only depend on values of $u_{n,m}$ with $m = 0$. More precisely, $v_{0,0}^{(i)}$ depends only on $u_{n,0}$ with $0 \leqslant n \leqslant (i+1)$, is homogeneous of degree $1 - 2i$ in these variables, and only depends on these variables through the combinations $u_{2,0} - u_{0,0}, u_{3,0} - u_{1,0}, \ldots, u_{i+1,0} - u_{i-1,0}$.

As in the case of continuum KdV an infinite sequence of conservation laws can be obtained starting from the ϵ expansion of a single 'conservation law' written in terms of u and $\tilde{u}$. It is straightforward to check that if we define

$$F = -\ln(\tilde{u}_{0,0} - u_{0,1}), \qquad G = \ln(\tilde{u}_{0,0} - u_{1,0})$$

then

$$(S_l - I)G + (S_k - I)F = \ln \frac{(\tilde{u}_{0,0} - u_{0,1})(\tilde{u}_{0,1} - u_{1,1})}{(\tilde{u}_{1,0} - u_{1,1})(\tilde{u}_{0,0} - u_{1,0})} = 0.$$

It only remains to explicitly expand F and G in powers of ϵ. We have

$$F = -\ln\left(u_{1,0} - u_{0,1} + \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i \right) = -\ln(u_{1,0} - u_{0,1}) - \ln\left(1 + \frac{1}{u_{1,0} - u_{0,1}} \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i \right),$$

$$G = \ln\left(\sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i\right) = \ln\epsilon - \ln(u_{0,0} - u_{2,0}) + \ln\left(1 + \frac{1}{v_{0,0}^{(1)}}\sum_{i=1}^{\infty} v_{0,0}^{(i+1)}\epsilon^i\right). \tag{10}$$

Writing $F = \sum_{i=0}^{\infty} F_i\epsilon^i$, $G = \ln\epsilon + \sum_{i=0}^{\infty} G_i\epsilon^i$ and introducing the notation

$$A_i = S_k^i\left(\frac{1}{u_{0,0} - u_{2,0}}\right), \qquad i = 0, 1, 2, \ldots, \qquad B = \frac{1}{u_{1,0} - u_{0,1}},$$

we obtain

$$\begin{cases} F_0 = \ln B \\ G_0 = \ln A_0, \end{cases}$$
$$\begin{cases} F_1 = -BA_0 \\ G_1 = A_0A_1, \end{cases}$$
$$\begin{cases} F_2 = -A_0^2A_1B + \frac{1}{2}A_0^2B^2 \\ G_2 = A_0A_1^2A_2 + \frac{1}{2}A_0^2A_1^2, \end{cases} \tag{11}$$
$$\begin{cases} F_3 = -A_0^2A_1^2A_2B - A_0^3A_1^2B + A_0^3A_1B^2 - \frac{1}{3}A_0^3B^3 \\ G_3 = A_0A_1^2A_2^2A_3 + A_0^2A_1^3A_2 + A_0A_1^3A_2^2 + \frac{1}{3}A_0^3A_1^3 \end{cases} \quad \text{etc.}$$

For $i > 0$, F_i is homogeneous of degree $2i$ in the variables $A_0, A_1, \ldots, A_{i-1}, B$ and G_i is homogeneous of degree $2i$ in the variables $A_0, A_1, \ldots, A_i$. We note that in all the conservation laws that we have computed, the G components are a sum of terms of the form '$p_k p_{k+1}$'. Thus, for example, in G_3 there are terms of the form $A_0^2A_1^3A_2$ (take $p_k = A_0^2A_1$) and $A_0A_1^3A_2^2$ (take $p_k = A_0^1A_1^2$) but no terms of the form $A_0^3A_1^2A_2$ or $A_0^2A_1^2A_2^2$.

Thus we see how expansion of the Bäcklund transformation around the point $\theta = \alpha$ yields an infinite sequence of conservation laws on the horizontal line for dKdV. Expansion around $\theta = \beta$ yields an infinite sequence of conservation laws on the vertical line. We have not yet given a proof that the conservation laws we have found by the Gardner method are all nontrivial, but this will emerge from analysis of the continuum limit in section 5.

4. The symmetry method for dKdV

We now turn to the symmetry method for generating dKdV conservation laws, as proposed by Rasin and Hydon [28]. The method proceeds by the repeated application of a certain symmetry to a certain basic conservation law.

We start by reviewing the necessary theory [16, 27]. An infinitesimal symmetry for the quad-graph equation (2) is an infinitesimal transformation of the form

$$\begin{aligned} u_{0,0} &\to \hat{u}_{0,0} = u_{0,0} + \epsilon Q(k, l, u, \mathbf{a}) + O(\epsilon^2), \\ \mathbf{a} &\to \hat{\mathbf{a}} = \mathbf{a} + \epsilon\xi(\mathbf{a}) + O(\epsilon^2), \end{aligned} \tag{12}$$

that maps solutions into solutions. The functions Q and ξ are called the characteristics of the symmetry; the function Q depends on finitely many shifts of $u_{0,0}$. The symmetry is often written in the form

$$X = Q\frac{\partial}{\partial u_{0,0}} + \xi\cdot\frac{\partial}{\partial\mathbf{a}}, \tag{13}$$

which is also referred to as the symmetry generator. By shifting (12) in the k and l directions we obtain that under the symmetry

$$u_{i,j} \to \hat{u}_{i,j} = u_{i,j} + \epsilon S_k^i S_l^j Q + O(\epsilon^2),$$

for every $i, j \in \mathbb{Z}$. Expanding (2) to first order in ϵ yields the symmetry condition

$$\hat{X}(P) = 0 \qquad \text{whenever (2) holds,}$$

where $\hat{X}$ is the 'prolonged' generator:

$$\hat{X} = \sum_{i,j} S_k^i S_l^j (Q) \frac{\partial}{\partial u_{i,j}} + \xi \cdot \frac{\partial}{\partial \mathbf{a}}. \tag{14}$$

In what follows, we will not distinguish between the generator of a symmetry and its prolongation, typically from the context it is clear which one is meant.

X_m is a *master symmetry* [9, 10, 25, 30] for the symmetry X if it satisfies

$$[X_m, X] \neq 0, \qquad [[X_m, X], X] = 0. \tag{15}$$

Here $[\cdot, \cdot]$ denotes the commutator.

Given a conservation law, a new conservation law can be obtained by applying a symmetry generator. This is possible since the (prolonged) symmetry generator commutes with shift operators, i.e.,

$$[X, S_k] = 0, \qquad [X, S_l] = 0.$$

So if F and G are components of a conservation law, i.e.

$$(S_k - I)F + (S_l - I)G = 0|_{P=0},$$

then by applying the symmetry generator we obtain

$$0 = X((S_k - I)F) + X((S_l - I)G)|_{P=0} = (S_k - I)X(F) + (S_l - I)X(G)|_{P=0}.$$

Thus

$$F_{\text{new}} = X(F), \qquad G_{\text{new}} = X(G)$$

is also a conservation law.

Known symmetry generators for dKdV (1) include

$$\begin{aligned} X_0 &= \frac{1}{u_{1,0} - u_{-1,0}} \frac{\partial}{\partial u_{0,0}}, & Y_0 &= \frac{1}{u_{0,1} - u_{0,-1}} \frac{\partial}{\partial u_{0,0}}, \\ X &= \frac{k}{u_{1,0} - u_{-1,0}} \frac{\partial}{\partial u_{0,0}} - \partial_\alpha, & Y &= \frac{l}{u_{0,1} - u_{0,-1}} \frac{\partial}{\partial u_{0,0}} - \partial_\beta. \end{aligned} \tag{16}$$

(The rationale for the notation here should become clear later.) Known conservation laws include

$$\begin{aligned} F &= \ln(u_{0,1} - u_{-1,0}), & G &= -\ln(u_{1,0} - u_{-1,0}), \\ \bar{F} &= \ln(u_{0,1} - u_{0,-1}), & \bar{G} &= -\ln(u_{1,0} - u_{0,-1}), \\ \tilde{F} &= kF + l\bar{F}, & \tilde{G} &= kG + l\bar{G}. \end{aligned} \tag{17}$$

(The first of these is the leading order conservation law found by the Gardner method in the previous section. The second is the leading order conservation law found using the Gardner method expanding the Bäcklund transformation around $\theta = \beta$.)

The symmetry method for constructing an infinite sequence of conservation laws is as follows:

Theorem 1. *The dKdV equation has an infinite number of nontrivial conservation laws on the horizontal line, generated by repeated application of the symmetry X to the conservation law with components (F, G), and an infinite number on the vertical line, generated by repeated application of the symmetry Y to the conservation law with components $(\bar{F}, \bar{G})$.*

Proof. We look at the horizontal case, the vertical case is similar.

It is known [27] that equation (1) has infinite hierarchies of symmetries in both the k and l directions. The symmetries in the k direction can be obtained by repeatedly commuting X with X_0:

$$X_1 = [X, X_0], \qquad X_2 = [X, X_1], \ldots .$$

(Similarly the symmetries in the l direction are obtained by commuting Y with Y_0.) The generator of X_n is $Q_n \frac{\partial}{\partial u_{0,0}}$ where the characteristic Q_n depends (at most) on the $2n + 3$ variables $u_{-n-1,0}, u_{-n,0}, \ldots, u_{n,0}, u_{n+1,0}$. In particular, note that Q_n does not depend on k; X does, but since X is a master symmetry for X_0, the k-dependence disappears on forming the necessary commutators. All the X_n are different (linearly independent).

Let us denote

$$F_n = X^n(F), \qquad G_n = X^n(G), \qquad n = 0, 1, 2, \ldots .$$

From the forms of X and G it follows that G_n depends (at most) on k and the $2n + 3$ variables $u_{-n-1,0}, u_{-n,0}, \ldots, u_{n,0}, u_{n+1,0}$. It is homogeneous of order $-2n$ in the u variables. From this homogeneity it follows that so long as the G_n are nontrivial then they will also not be dependent. Furthermore the G_n are valued on the horizontal line. By the lemma of section 2 it follows that if the conservation law G_n is trivial we must have $G_n = (S_k - I) f_n$ for some function f_n valued on the line, and therefore $E(G_n) = 0$, where E is the Euler operator. Thus we can prove nontriviality by verifying that $E(G_n) \neq 0$. We now show that

$$E(G_n) = (S_k - S_k^{-1}) Q_n, \qquad n = 0, 1, 2, \ldots, \tag{18}$$

where Q_n is the characteristic of the symmetry generator X_n introduced above. Since all the necessary quantities that appear are valued on the horizontal line, we drop the vertical index on values of u in all the calculations that follow, and denote the horizontal shift simply as S (instead of S_k).

First we verify (18) in the case $n = 0$. Using $G_0 = -\ln(u_1 - u_{-1})$ and $Q_0 = \frac{1}{u_1 - u_{-1}}$ we obtain

$$E(G_0) = S^{-1}\frac{\partial G_0}{\partial u_1} + S\frac{\partial G_0}{\partial u_{-1}} = -\frac{1}{u_0 - u_{-2}} + \frac{1}{u_2 - u_0} = (S - S^{-1})Q_0$$

as desired.

Now assume that (18) is true for $n = r - 1$. We have

$$\begin{aligned}
E(G_r) &= E(X(G_{r-1})) \\
&= E\left(\sum_{i=-r}^{r} (S^i Q)\frac{\partial G_{r-1}}{\partial u_i}\right) \\
&= \sum_{j=-r-1}^{r+1} \sum_{i=-r}^{r} S^{-j} \frac{\partial}{\partial u_j}\left[(S^i Q)\frac{\partial G_{r-1}}{\partial u_i}\right] \\
&= \sum_{j=-r-1}^{r+1} \sum_{i=-r}^{r} \left[\left(S^{-j}\frac{\partial (S^i Q)}{\partial u_j}\right)\left(S^{-j}\frac{\partial G_{r-1}}{\partial u_i}\right) + (S^{i-j}Q)S^{-j}\left(\frac{\partial^2 G_{r-1}}{\partial u_i \partial u_j}\right)\right] \\
&= \sum_{j=-r-1}^{r+1} \sum_{i=-r}^{r} \left[\left(S^{i-j}\frac{\partial Q}{\partial u_{j-i}}\right)\left(S^{-j}\frac{\partial G_{r-1}}{\partial u_i}\right) + (S^{i-j}Q)S^{-j}\left(\frac{\partial^2 G_{r-1}}{\partial u_i \partial u_j}\right)\right].
\end{aligned} \tag{19}$$

Here Q denotes the characteristic of the symmetry X. Note that in the second term in (19), the terms with $j = -r - 1$ and $j = r + 1$ do not contribute. In fact the second term is precisely

$X(E(F_{r-1}))$, as the following calculation shows:

$$
\begin{aligned}
X(E(G_{r-1})) &= X\left(\sum_{j=-r}^{r} S^{-j}\frac{\partial G_{r-1}}{\partial u_j}\right)\\
&= \sum_{j=-r}^{r}\sum_{k=-r-j}^{r-j}(S^kQ)\frac{\partial}{\partial u_k}\left(S^{-j}\frac{\partial G_{r-1}}{\partial u_j}\right)\\
&= \sum_{j=-r}^{r}\sum_{k=-r-j}^{r-j}(S^kQ)S^{-j}\frac{\partial^2 G_{r-1}}{\partial u_j\partial u_{k+j}}.
\end{aligned}
$$

The last expression is seen to be the same as the second term in (19) by replacing the summation variable k by $i = k + j$. With regard to the first term in (19), note that Q only depends on u_1 and u_{-1} and so there are only nonzero contributions when $j = i + 1$ or $j = i - 1$. Thus we have

$$
\begin{aligned}
E(G_r) &= \sum_{i=-r}^{r}\left(S^{-1}\frac{\partial Q}{\partial u_1}\right)\left(S^{-i-1}\frac{\partial G_{r-1}}{\partial u_i}\right) + \sum_{i=-r}^{r}\left(S\frac{\partial Q}{\partial u_{-1}}\right)\left(S^{-i+1}\frac{\partial G_{r-1}}{\partial u_i}\right) + X(E(G_{r-1}))\\
&= \left(S^{-1}\frac{\partial Q}{\partial u_1}\right)S^{-1}E(G_{r-1}) + \left(S\frac{\partial Q}{\partial u_{-1}}\right)SE(G_{r-1}) + X(E(G_{r-1}))\\
&= (S^{-1} - S)\left(\frac{\partial Q}{\partial u_1}E(G_{r-1})\right) + X(E(G_{r-1})).
\end{aligned}
$$

In the last line we have used the fact that

$$
\frac{\partial Q}{\partial u_{-1}} = -\frac{\partial Q}{\partial u_1}.
$$

Now we use the induction hypothesis $E(G_{r-1}) = (S - S^{-1})Q_{r-1}$. Since shift operators commute with any prolonged symmetry operator we obtain at once that

$$
\begin{aligned}
E(G_r) &= (S - S^{-1})\left(-\frac{\partial Q}{\partial u_1}(S - S^{-1})Q_{r-1} + X(Q_{r-1})\right)\\
&= (S - S^{-1})\left(-(SQ_{r-1})\frac{\partial Q}{\partial u_1} - (S^{-1}Q_{r-1})\frac{\partial Q}{\partial u_{-1}} + X(Q_{r-1})\right)\\
&= (S - S^{-1})(-X_{r-1}(Q) + X(Q_{r-1})).
\end{aligned}
$$

But since $X_r = [X, X_{r-1}]$, $Q_r = X(Q_{r-1}) - X_{r-1}(Q)$. Thus we have $E(G_r) = (S - S^{-1})Q_r$, as required, providing the induction step for our claim that $E(G_n) = (S - S^{-1})Q_n$ for all n. In particular $E(G_n) \neq 0$, giving nontriviality of the conservations laws with components F_n, G_n. □

5. A continuum limit

In this section we consider a continuum limit of (1). By replacing

$$
u_{i,j} = u(x + ih, t + jh), \qquad \alpha = \alpha(h), \qquad \beta = \beta(h),
$$

and dividing equation (1) by h^2 we obtain

$$
\frac{(u(x+h,t+h) - u(x,t))(u(x+h,t) - u(x,t+h))}{h^2} = \frac{\alpha(h) - \beta(h)}{h^2}. \tag{20}
$$

Taking the limit of (20) as $h \to 0$ we obtain

$$u_x^2 - u_t^2 = C, \tag{21}$$

where $C = \lim_{h\to 0} \frac{\alpha(h)-\beta(h)}{h^2}$, assuming the limit exists. This continuum limit of dKdV is not the limit in which the potential KdV equation is recovered; instead we have obtained the eikonal equation (21). This turns out, however, to be an advantage as its symmetry analysis is quite straightforward.

It is clear that the applying limiting procedure to symmetries and conservation laws for (1) gives symmetries and conservation laws for (21). For example, the limits of the symmetries in (16) are

$$\begin{aligned} X_0 &= \frac{1}{2u_x}\partial_u, & Y_0 &= \frac{1}{2u_t}\partial_u, \\ X &= \frac{x}{2u_x}\partial_u + \partial_C, & Y &= \frac{t}{2u_t}\partial_u + \partial_C, \end{aligned} \tag{22}$$

and the limits of the conservation laws in (17) are

$$\begin{aligned} F &= \ln(u_x + u_t), & G &= -\ln(u_x), \\ \bar{F} &= \ln(u_t), & \bar{G} &= -\ln(u_x + u_t), \\ \tilde{F} &= xF + t\bar{F}, & \tilde{G} &= xG + t\bar{G}. \end{aligned}$$

The limit of the symmetry construction of conservation laws is as follows:

Theorem 2. *Equation (21) has an infinite number of distinct, nontrivial conservation laws generated by repeated application of the symmetry X to the conservation law with components (F, G). Writing $F_n = X^n(F), G_n = X^n(G)$ we find $G_n = \frac{1}{u_x^{2n}}$ for $n \geqslant 1$ (up to addition of a trivial conservation law and rescaling).*

Proof. We have

$$\begin{aligned} G_1 = X(G_0) &= -\left(\frac{x}{2u_x}\right)_x \partial_{u_x}(\ln u_x) \\ &= -\frac{1}{2u_x^2} + \frac{xu_{xx}}{2u_x^3} \\ &= -\left(\frac{x}{4u_x^2}\right)_x - \frac{1}{4u_x^2}. \end{aligned}$$

The first term on the RHS is the 'G' component of a trivial conservation law; the second is a multiple of $\frac{1}{u_x^2}$, proving the result for $n = 1$. For $n \geqslant 1$ we have

$$\begin{aligned} X\left(\frac{1}{u_x^{2n}}\right) &= \left(\frac{x}{2u_x}\right)_x \partial_{u_x}\left(\frac{1}{u_x^{2n}}\right) \\ &= -\frac{n}{u_x^{2n+2}} + \frac{nxu_{xx}}{u_x^{2n+3}} \\ &= -\left(\frac{nx}{(2n+2)u_x^{2n+2}}\right)_x - \frac{n(2n+1)}{(2n+2)u_x^{2n+2}}. \end{aligned}$$

The first term on the RHS is the 'G' component of a trivial conservation law and the second is a multiple of $\frac{1}{u_x^{2(n+1)}}$. Thus the required form of G_n is established by induction. G_n is evidently not the x-derivative of a function of u and its x-derivatives and thus the conservations laws with components F_n, G_n are all nontrivial and distinct. □

Since it is impossible that distinct, nontrivial conservation laws be a continuum limit of conservation laws that are equivalent or trivial; this furnishes an alternative proof that the conservation laws constructed by the symmetry method in the discrete case are distinct and nontrivial.

For completeness we give a formula for F_n for $n \geqslant 1$. From the definition of a conservation law for (21) we have

$$\frac{\partial F_n}{\partial x} + \frac{\partial G_n}{\partial t} = 0 \tag{23}$$

on solutions of (21). Look for F_n as a function of u_x alone. We then need

$$F_n'(u_x)u_{xx} - \frac{2nu_{xt}}{u_x^{2n+1}} = 0.$$

But any solution of (21) also satisfies $u_x u_{xx} - u_t u_{tx} = 0$. Thus

$$F_n'(u_x) = \frac{2nu_{xt}}{u_x^{2n+1}u_{xx}} = \frac{2n}{u_x^{2n}u_t} = \frac{2n}{u_x^{2n}\sqrt{u_x^2 - C}},$$

and

$$F_n(u_x) = \int \frac{2n}{u_x^{2n}\sqrt{u_x^2 - C}}\,\mathrm{d}u_x.$$

These integrals can be computed using $F_1(u_x) = \frac{\sqrt{u_x^2 - C}}{Cu_x}$ and the recursion

$$F_n(u_x) = \frac{2n}{(2n-1)C}\left(F_{n-1}(u_x) + \frac{\sqrt{u_x^2 - C}}{(2n-1)u_x^{2n-1}}\right), \qquad n > 1.$$

This results in an expression for $F_n(u_x)$ that is the product of a rational function of u_x and C with $\sqrt{u_x^2 - C}$. $\sqrt{u_x^2 - C}$ can then be replaced by u_t, and, if desired, all occurrences of C can be replaced by $u_x^2 - u_t^2$, giving a rational function of u_x and u_t.

Theorem 3. *The continuum limit, in the sense we have described, of the conservation laws for dKdV constructed using the Gardner transformation coincides with the continuum limit of those constructed by the symmetry method.*

Proof. In the limit of small h, equation (8) gives

$$v_{0,0}^{(1)} \sim -\frac{h}{2u_x}.$$

Equation (9) gives

$$v_{0,0}^{(i)} \sim -C_{i-1}\left(\frac{h}{2u_x}\right)^{2i-1}, \qquad i = 2, 3, \ldots,$$

where C_n are the Catalan numbers,

$$C_n = \frac{(2n)!}{n!(n+1)!},$$

which satisfy the recursion [32]

$$C_0 = 1, \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}, \qquad n \geqslant 0.$$

Using these results in (10) gives

$$G_n \sim H_n\left(\frac{h}{2u_x}\right)^{2n}, \qquad n = 1, 2, \ldots,$$

where the numbers H_n are defined by the generating function

$$\sum_{n=1}^{\infty} H_n t^n = \ln\left(1 + \sum_{n=1}^{\infty} C_n t^n\right) = \ln\left(\frac{1-\sqrt{1-4t}}{2t}\right).$$

Here in the last equality we have used the standard expression for the generating function for the Catalan numbers [32]. Remarkably, there is a simple formula for the H_n:

$$H_n = \frac{n+1}{2n} C_n = \frac{(2n-1)!}{(n!)^2}, \qquad n = 1, 2, \ldots.$$

This can be proved from the fact that if we denote the generating function of the Catalan numbers by $c(t) = \frac{1-\sqrt{1-4t}}{2t}$, then

$$\log(c(t)) = \frac{c(t)-1}{2} + \int_0^t \frac{c(s)-1}{2s}\,\mathrm{d}s.$$

For our purposes, however, it is just necessary to observe that $H_n \neq 0$, so, to leading order in h, G_n is a nonzero multiple of $\frac{1}{u_x^{2n}}$. □

To take the limit of the various F_n in (11), we use the fact that each of the A_i behaves as $-\frac{h}{2u_x}$ while B behaves as $\frac{h}{u_x+u_t}$. Thus in the limit we obtain F_n as a rational function of u_x and u_t, in agreement with our previous conclusions.

To summarize, in this section we have computed a continuum limit of the conservation laws for dKdV found by the symmetry method and the Gardner transformation method. Since the continuum limits of the conservation laws are nontrivial, so are the original ones. Furthermore the fact that the limits of the two sets of conservation laws coincide strongly suggests that the two sets of conservation laws are identical, but we have not yet succeeded in proving this.

6. Concluding remarks

In this paper we have made substantial progress understanding conservation laws for the dKdV equation. We have presented two methods for constructing an infinite number of nontrivial, distinct conservation laws. The forms of the conservation laws are very similar, and we have seen that in a certain continuum limit they coincide, leading us to hypothesize that they are in fact equal.

In addition to proving the two sets of conservation laws are equal, much more remains to be done. There are other constructions of the conservation laws for continuum KdV, such as the Lenard recursion [23] and the method of Drinfeld-Sokolov [8], based upon the zero-curvature formulation. It is interesting to know if these have analogs for dKdV. We note that proving equivalence of the different constructions for continuum KdV is also nontrivial [33]. We mentioned in section 3 the curious fact that the 'G' components of the first few conservation laws constructed by the Gardner method all have the form of a sum of terms of the form $p_k p_{k+1}$. Sums of the form $\sum_k \psi_k \psi_{k+1}$ also appear as the discrete analog of the L_2 norm $\int \psi^2\,\mathrm{d}x$ in the context of the discrete Schrödinger equation upon which the inverse scattering theory for dKdV is built [5, 6].

Another completely open question is to understand the constraints on the dynamics of the dKdV equation that come about as a result of the infinite number of conservation laws. In the case of continuum KdV, the conservation laws give rise to bounds on Sobolev norms of the solution, thus preventing initially smooth data becoming non-smooth [19]. The conservation laws for dKdV discussed in this paper do not seem to be appropriate for this; maybe there are other conservation laws, or maybe some other understanding is appropriate.

Finally, a lot of the work in this paper can be generalized to other equations in the ABS classification, though there are numerous subtleties. The homogeneity argument, used in the proof that the conservation laws obtained by the symmetry method are nontrivial, breaks down, and for certain equations it seems that the continuum limits of the conservation laws are trivial. A paper on this subject is in preparation.

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